BÉZOUT'S THEOREM AND VARIETIES OF MINIMAL DEGREE

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Abstract. In this work we try to generalize Bézout's Theorem for curves, which was stated and applied in Talks 11 and 12. To this end, we follow a geometrical approach. We start by giving three equivalent and geometrical definitions of the degree of an arbitrary variety. Then, we state and prove a weak version of Bézout which does not take into account the notion of intersection multiplicity. After this, we state the full Bézout's Theorem and give some immediate consequences. Finally, as a nice application of Bézout's Theorem, we study curves of minimal degree, and state the classification theorem of varieties (of arbitrary dimension) of minimal degree.

1. Introduction

A key step in the proof of the classification of plane cubic curves (talks 11 and 12) was given by the following result.

Proposition 1.1. *Let k be an algebraically closed field, $C, C' \subset \mathbb{P}^2_k$ curves of degrees d, d', respectively, and which have no common components. Then, C and C' intersect exactly at $d \cdot d'$ points counted with multiplicity, i.e.*

$$\sum_{p \in C \cap C'} I_p(C, C') = d \cdot d'.$$

The main goal of this talk is to generalize this result to arbitrary varieties and to give some consequences. In order to do this, we will first define the degree of a general variety in Section 2. Then, we will state and prove Bézout's Theorem in Section 3. Finally, in Section 4 we will prove that rational normal curves are exactly the non-degenerate curves of minimal degree and we will state the classification of general varieties of minimal degree.

2. Degree of a variety

1. **Preliminary notions.** Throughout this script, we will work over an algebraically closed field of characteristic 0. In fact, by the Lefschetz principle, we might as well work on $\mathbb{C}$. We will make no further mention to the base field unless necessary. We start by recalling the concept of generality.

Definition 2.1. Let $X = \{X_p\}_{p \in Z}$ be a collection of objects (e.g. points, varieties, maps, etc.) parametrized by an irreducible variety Z. We say that *the general object X satisfies property P* if the set $\{p \in Z | X_p \text{ satisfies } P\}$ contains an open (and hence dense) subset of Z.

Since we will approach the concept of degree from a geometrical point of view, we will also need a characterization of dimension in geometrical terms.

Proposition 2.2. *(Characterizations of dimension) Let $X \subseteq \mathbb{P}^n$ be an irreducible variety. The following are equivalent:*

(a) *The (Krull) dimension of X is k.*
(b) *A general $(n-k)$-plane in $\mathbb{P}^n$ intersects X at a finite set of points.*
(c) *k is the smallest integer such that a general $(n-k-1)$-plane in $\mathbb{P}^n$ is disjoint from X.*
(d) *Projecting from a general $(n-k-1)$-plane defines a finite surjective map $X \to \mathbb{P}^k$.*
(e) *Projecting from a general $(n-k-2)$-plane defines a map $X \to \mathbb{P}^{k+1}$ which is birrational onto its image $\overline{X}$ and this image is a hypersurface in $\mathbb{P}^{k+1}$.*

For a proof, see [Har92] Lecture 11.

Remark 2.3. In (b), we mean that the set of points in $\mathbb{G}(n-k,n)$ that intersect X at a finite set of points contains an open subset of $\mathbb{G}(n-k,n)$ (recall that Grassmanians are irreducible varieties).

2. **Definitions of degree.** The previous proposition allows us to give a first definition of the degree of a variety.

Definition 2.4. Let $X \subseteq \mathbb{P}^n$ be an irreducible k-dimensional variety. The *degree of X*, denoted $\deg(X)$, is defined to be $\#(\Omega \cap X)$ where Ω is a general $(n-k)$-plane in $\mathbb{P}^n$. The degree of a reducible variety is defined to be the sum of the degrees of its irreducible components.

Remark 2.5. The degree is well-defined. One way to see this is to look at the projection

$$\Omega^{(k)}(X) \to \mathbb{G}(n-k,n),$$

where $\Omega^{(k)}(X) = \{(\Lambda, p) : \Lambda \in \mathbb{G}(n-k,n), p \in \Lambda \cap X\}$ is the universal k-fold hyperplane section of X. Then, use Proposition 7.16 in [Har92] to show that a general fiber has a fixed finite number of points. Notice that the points of the fiber over Ω correspond to the points $\Omega \cap X$.

A bit of an alternative overkill approach would be to prove that any $(n-k)$-plane Ω that intersects X transversely gives the same degree and use the fact that a general $(n-k)$-plane Ω intersects X transversely (cf. Lemma 3.9).

Our goal now is to give two further equivalent definitions of degree.

Proposition 2.6. *The cardinality of a general fiber of the projection $\pi_\Lambda : X \to \mathbb{P}^k$ from a general $(n-k-1)$-plane $\Lambda \subset \mathbb{P}^n$ (see Proposition 2.2) equals* $\deg(X)$.

Proof. Consider the continuous map

$$\mathbb{G}(n-k-1,n) \times \mathbb{P}^n \to \mathbb{G}(n-k,n)$$

which takes a pair (Λ, p) to the $(n-k)$-plane spanned by Λ and p. Of course, this is only defined on an open set of $\mathbb{G}(n-k-1,n) \times \mathbb{P}^n$. The preimage U of the set of $(n-k)$-planes that give the right degree of X will certainly contain an open subset of $\mathbb{G}(n-k-1,n)$. Consider an element $\Lambda \in U$ and the corresponding $(n-k)$-plane Ω containing Λ that gives the right degree of X. Then, Ω intersects $\mathbb{P}^k$ at a single point p. The fiber of π_Λ over p consists exactly of $\Omega \cap X$.

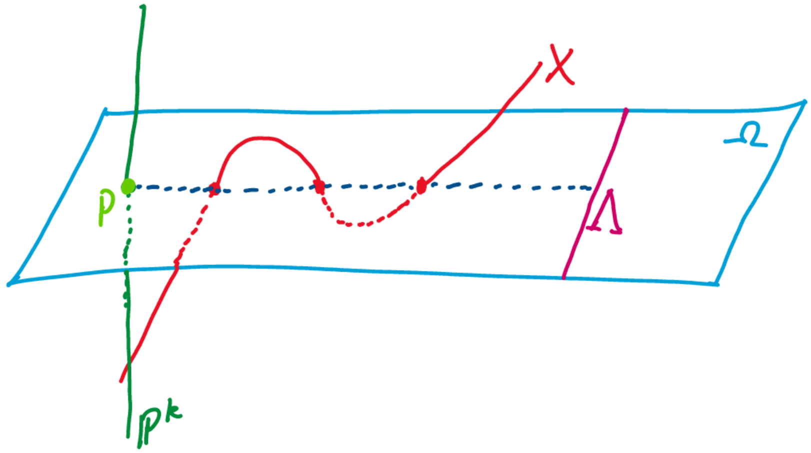


FIGURE 1. Proof of Proposition 2.6.

□

Remark 2.7. This proof shows both the equivalence and the well-definedness of this alternative definition.

Proposition 2.8. *Let $X \subset \mathbb{P}^n$ be an irreducible k-dimensional variety. Then:*

(a) If $X \subset \mathbb{P}^n$ is a hypersurface, then $X = V(f)$ for an irreducible homogeneous polynomial f and $\deg(X) = \deg(f)$.

(b) If $k < n-1$, then projecting from a general $(n-k-2)$-plane defines a map $\pi_\Gamma : X \to \mathbb{P}^{k+1}$ which is birrational onto the hypersurface $\overline{X} = \pi_\Gamma(X) \subset \mathbb{P}^{k+1}$ (see Proposition 2.2). Then, $\deg(X) = \deg(f)$*, where f is the irreducible polynomial that defines $\overline{X}$.*

Proof. For (a), write $f(Z_0, \dots, Z_n) = \sum a_I Z_I$. Pick a general line $L \subset \mathbb{P}^n$. By performing a projective transformation, assume $L = \{Z_2 = \dots = Z_n = 0\}$. Then $X \cap L$ consists of points $[Z_0 : Z_1 : 0 : \dots : 0]$ satisfying $f(Z_0, Z_1, 0, \dots, 0) = \sum_i a_i Z_0^i Z_1^{d-i} = 0$, which in general has d distinct solutions.[1]

For (b), as in the proof of Proposition 2.6, consider the continuous (on an open set) map

$$\mathbb{G}(n-k-2, n) \times \mathbb{P}^n \to \mathbb{G}(n-k-1, n).$$

The preimage V of the set of $(n-k-1)$-planes that give the right degree of X certainly contains an open subset of $\mathbb{G}(n-k-2, n)$. Consider an element $\Gamma \in V$ and its corresponding $(n-k-1)$-plane Λ that contains Γ and gives the right degree of X. So far we have projections $\pi_\Gamma : X \to \mathbb{P}^k$, $\pi_\Lambda : X \to \mathbb{P}^{k+1}$. We can choose $\mathbb{P}^k \subset \mathbb{P}^{k+1}$. Now, Λ intersects $\mathbb{P}^{k+1}$ at a point p. We can write $\pi_\Lambda = \pi_p \circ \pi_\Gamma$. In particular, the general fiber $\pi_\Lambda^{-1}(q)$ (whose cardinality equals $\deg(X)$ by Proposition 2.6) is the same as $\pi_\Gamma^{-1}(\pi_p^{-1}(q))$. But $\pi_p^{-1}(q)$ is the intersection of $\overline{X}$ with the general line $\overline{p, q}$ and since π_Γ is generally one-to-one, we are done by (a).

[1]By Lemma 3.9, a general line intersects X transversely. By the proof of 3.11 the polynomial has d distinct solutions.

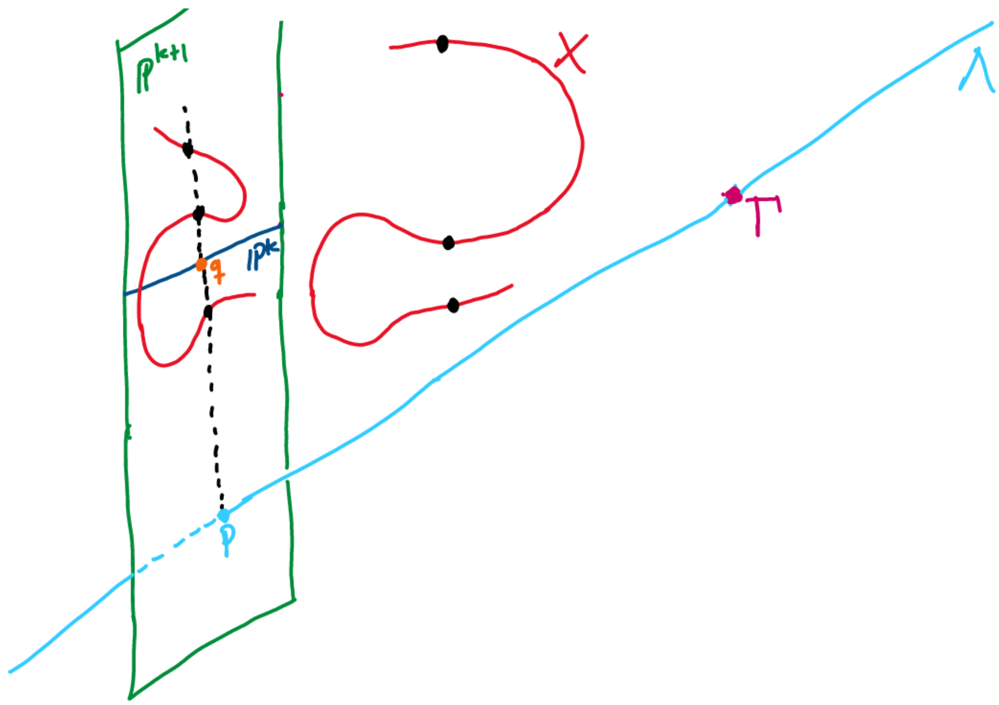


FIGURE 2. Proof of Proposition 2.8(b).

□

Remark 2.9. Again, this proof shows both the equivalence and the well-definedness of this alternative definition.

We summarize these definitions as follows:

Definition 2.10. Let $X \subset \mathbb{P}^n$ be an irreducible k-dimensional variety. Let Ω, Λ and Γ be a general $(n-k)$-plane, $(n-k-1)$-plane and $(n-k-2)$-plane. Then, the *degree of* X $\deg(X)$ can be defined to be either

(i) $\#(\Omega \cap X)$,
(ii) $\#\pi_\Lambda^{-1}(p)$ for a general $p \in \mathbb{P}^k$, or
(iii) The degree of the irreducible polynomial that defines the hypersurface $\pi_\Gamma(X) \subset \mathbb{P}^{k+1}$.

If X is not irreducible, the degree is defined to be the sum of the degree of its irreducible components.

Remark 2.11. An algebraic approach to the concept of degree is also possible through the notion of Hilbert Polynomials (cf. for example Chapter 18 in [Har92] or Section I.7 in [Har77]). This has the downside of being a cumbersome definition, but it makes life easier when proving general results about degree.

3. **Some examples and properties.** We now give some examples and easy results concerning the degree.

Example 2.12. (1) The degree of a point p in $\mathbb{P}^n$ is 1. Indeed, a point is a 0-dimensional irreducible variety, so in order to obtain its degree, we have to intersect with a general n-plane, i.e. the whole $\mathbb{P}^n$. This intersection only consists of the point p, so $\deg(p) = 1$.

(2) Slightly more generally, the degree of a finite set of points $\{p_1, \ldots, p_d\}$ is d.
(3) The degree of a linear variety is 1. Indeed, a linear k-dimensional variety can be described by $n-k$ equations. When intersecting with a general $(n-k)$-dimensional variety, we obtain n linear equations in n variables, which in general have a unique solution.
(4) The degree of a hypersurface $X = V(f)$, with f irreducible, is the degree of f.

Remark 2.13. The degree depends on the embedding into the ambient space. For example, any linear embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ has degree 1. However, the rational normal curve $\nu_n(\mathbb{P}^1)$, where

$$\begin{aligned} \nu_n : \mathbb{P}^1 &\to \mathbb{P}^n \\ [X_0 : X_1] &\mapsto [X_0^n : X_0^{n-1}X_1 : \ldots : X_1^n], \end{aligned}$$

is isomorphic to $\mathbb{P}^1$ but has degree n. Indeed, consider a general hyperplane $a_0 Z_0 + \ldots + a_n Z_n = 0$. By intersecting such a hyperplane with $\nu_n(\mathbb{P}^1)$ we obtain an equation $a_0 X_0^n + a_1 X_0^{n-1} X_1 + \ldots + a_d X_1^n = 0$. Since this is a degree n polynomial, it has in general n distinct solutions.

Remark 2.14. The degree is invariant under the action of $\mathrm{Aut}(\mathbb{P}^n)$.

Remark 2.15. If $X \subset \mathbb{P}^n$ is an irreducible k-dimensional variety and L is a general $(n-l)$-plane with $l \leq k$, then $\deg(X) = \deg(X \cap L)$.

Proof. Take the preimage of an appropriate open set $U \subset \mathbb{G}(n-k, n)$ under the map

$$\mathbb{G}(n-l, n) \times \mathbb{G}(n-k+l, n) \to \mathbb{G}(n-k, n).$$

□

3. Bézout's Theorem

We will state two versions of Bézout's Theorem and prove the weaker one. We will need a few definitions first.

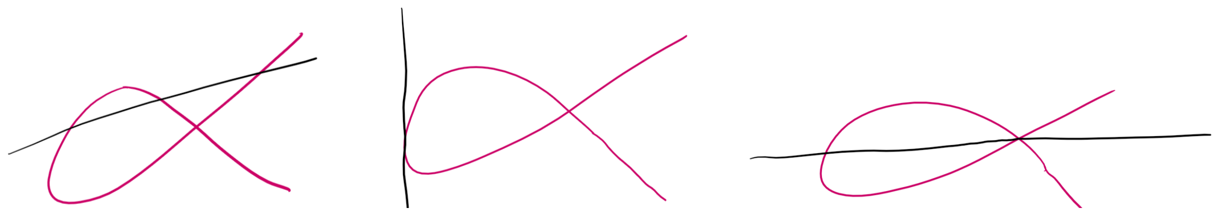
1. **Some preliminary definitions.**

Definition 3.1. Let $X, Y \subseteq \mathbb{P}^n$ be two varieties and let $X \cap Y = \bigcup_i Z_i$ be a decomposition of the intersection into irreducible varieties. Then, we say that X and Y intersect

(1) *transversely at* $p \in X \cap Y$ if both X and Y are smooth at p and $T_pX + T_pY = T_p\mathbb{P}^n$.
(2) *transversely* if they intersect transversely at every $p \in X \cap Y$.
(3) *generically transversely* if for every i, X and Y intersect transversely at a general point $p \in Z_i$.

Remark 3.2. If X and Y have pure dimension and $\dim(X) + \dim(Y) = n$, then conditions (2) and (3) are equivalent. Clearly (2) $\Rightarrow$ (3). For the converse, notice that because of the dimension condition, for a general $p \in Z_i$ we have $T_pX \cap T_pY = \{p\}$. Hence $\dim(Z_i) \leq \dim(T_pZ_i) \leq \dim(T_pX \cap T_pY) = 0$. This implies that each Z_i is consists only of one point, so when speaking of a general point of Z_i, we are just referring to the only point of Z_i. In particular, $X \cap Y$ consists of a finite number of points.

Example 3.3. In the following, "TI" stands for transverse intersection and "GTI" stands for generically transverse intersection.



(A) TI: yes, GTI: yes (B) TI: no, GTI: no (C) TI: no, GTI: no

FIGURE 3. Examples with a nodal cubic and a line

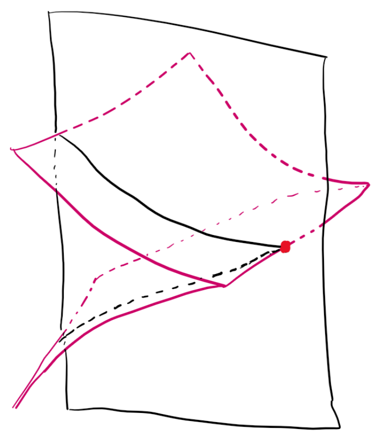


FIGURE 4. Example with a surface and a plane in $\mathbb{P}^3$. TI: no, GTI: yes

Definition 3.4. Let $X, Y \subseteq \mathbb{P}^n$ be varieties of pure dimension. We say that they *intersect properly* if every irreducible component of $X \cap Y$ has dimension $\dim(X) + \dim(Y) - n$.

Proposition 3.5. *If $X, Y \subseteq \mathbb{P}^n$ intersect generically transversely, then they also intersect properly.*

Proof. Exercise 1. □

2. **Statement and proof of the weak version.** We are now ready to state the first version of Bézout's Theorem.

Theorem 3.6. *(Weak Bézout) Let $X, Y \subseteq \mathbb{P}^n$ be varieties of pure dimension. If $\dim(X) + \dim(Y) \geq n$ and they intersect generically transversely, then*

$$\deg(X \cap Y) = \deg(X) \cdot \deg(Y) \tag{1}$$

Remark 3.7.
(1) If $\dim(X) + \dim(Y) = n$, by Remark 3.2 the theorem states that $X \cap Y$ consists of $\deg(X) \cdot \deg(Y)$ points.
(2) The hypotheses are sharp:
   (i) The base field must be algebraically closed: for example if it does not have a square root of -1, then $X = V(y), Y = V(yz - x^2 - z^2) \subset \mathbb{P}^2$ do not satisfy (1).

(ii) This does not work on the affine space: for example take two parallel distinct lines in $\mathbb{A}^2$.
(iii) Pure dimension: $X = V(x, y) \cup V(y - z)$ and $Y = V(x - z)$ in $\mathbb{P}^2$ do not satisfy (1).
(iv) $\dim(X) + \dim(Y) \geq n$: take two disjoint lines in $\mathbb{P}^3$.
(v) Generically transverse intersection: for example $X = V(x^2 - yz), Y = V(y) \subset \mathbb{P}^2$ intersect only at one point.

In order to prove Theorem 3.6, we will need a bunch of lemmas. We begin by stating (a weak version of) Bertini's Theorem, a proof of which can be found in [Har92] Lecture 17.

Theorem 3.8. *(Bertini) Let $X \subset \mathbb{P}^n$ be a variety and $H \subset \mathbb{P}^n$ a general hyperplane. Then $(X \cap H)_{sing} = X_{sing} \cap H$.*

Lemma 3.9. *Let $X \subset \mathbb{P}^n$ be a k-dimensional variety. Then, a general $(n-k)$-plane Ω intersects X transversely.*

Proof. A general $(n-k)$-plane Ω is the intersection of k general hyperplanes $H_1, \dots, H_k$. Since Ω intersects X at a finite number of points, $(X \cap \Omega)_{\text{sing}} = \varnothing$. Applying Bertini's Theorem repeatedly yields:

$$\begin{aligned}\varnothing &= (X \cap \Omega)_{\text{sing}} = (X \cap H_1 \cap \ldots \cap H_k)_{\text{sing}} \\ &= (X \cap H_1 \cap \ldots \cap H_{k-1})_{\text{sing}} \cap H_k = \ldots = X_{\text{sing}} \cap H_1 \cap \ldots \cap H_k \\ &= X_{\text{sing}} \cap \Omega.\end{aligned}$$

Hence, a general Ω intersects X at smooth points. Now, the fact that the set of $(n-k)$-planes that are tangent to X is a proper subvariety of $\mathbb{G}(n-k, n)$ finishes the proof. □

Lemma 3.10. *Let $X, Y \subset \mathbb{P}^n$ be varieties of pure dimensions k, l respectively. Suppose they intersect generically transversely. Then, for a general $(n-k-l)$-plane Ω, $(X \cap \Omega)$ intersects Y transversely.*

Proof. By hypothesis $X \cap Y$ consists of irreducible components Z. In each of these components, there are open subsets in which X and Y are smooth and their intersection is transverse. By Lemma 3.9, a general $(n-k-l)$-plane intersects each Z transversely at points at which the intersection of X and Y is transverse. □

The following is an even weaker version of Bézout's Theorem.

Lemma 3.11. *Let $X \subseteq \mathbb{P}^n$ be a variety of pure dimension k and $\Lambda \subset \mathbb{P}^n$ an $(n-l)$-plane with $l \leq k$. If X and Λ intersect generically transversely, then*

$$\deg(X \cap \Lambda) = \deg(X) \tag{2}$$

Proof. Wlog assume X is irreducible. Also Wlog we can assume that $k = l$. Indeed, suppose $l < k$. Then, by Lemma 3.10 and Remark 2.15, for a general $(k-l)$-plane, we would have

$$\deg(X \cap \Lambda) = \deg((X \cap \Lambda) \cap \Omega) = \deg(X \cap (\Lambda \cap \Omega)) = \deg(X),$$

as wanted.

So suppose $k = l$. We will proceed by induction on $\text{codim}(X) = n - k$. Suppose $\text{codim}(X) = 1$. Then $X = V(f)$ for an irreducible homogeneous polynomial f. By performing a projective transformation, we can assume that $\Lambda = \{Z_2 = \ldots = Z_n = 0\}$. Then, the intersection $X \cap \Lambda$ consists of points $[Z_0 : Z_1 : 0 : \ldots : 0]$ such that $g(Z_0, Z_1) =$

$f(Z_0, Z_1, 0, \dots, 0) = \sum_i a_i Z_0^i Z_1^{d-i} = 0$. Using our third definition of degree, it will suffice to show that this polynomial has d distinct roots. Suppose g has a double root at $p = [b_0 : b_1]$. By the chain rule, we have

$$0 = \frac{\partial g}{\partial Z_0}(p) = \frac{\partial f}{\partial Z_0}(p),$$
$$0 = \frac{\partial g}{\partial Z_1}(p) = \frac{\partial f}{\partial Z_1}(p).$$

Hence, the line Λ will be tangent to X at p, contradiction.

Now, suppose $\operatorname{codim}(X) > 1$. Pick an $(n-k-2)$-plane $\Gamma \subset \Lambda$ that does not intersect X. Consider the projection $\pi_\Gamma : X \to \mathbb{P}^{k+1}$. Then, the points of $(X \cap \Lambda)$ correspond bijectively to points in $\pi_\Gamma(X) \cap (\Lambda \cap \mathbb{P}^{k+1})$, which is a transverse intersection of a hypersurface and a line, and thus we are reduced to the previous case.

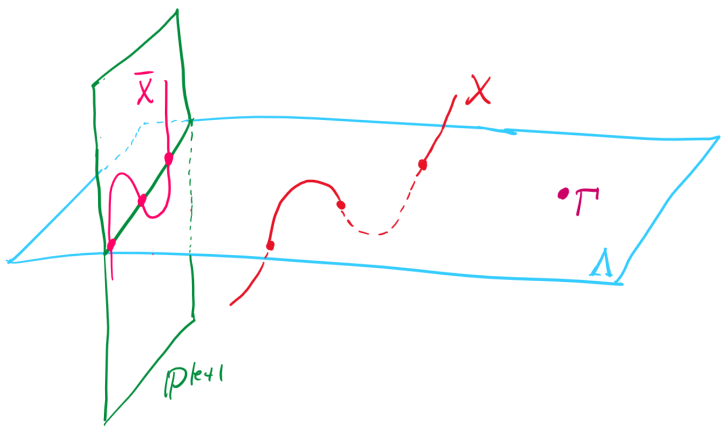


FIGURE 5. Proof of Lemma 3.11.

□

We will also use the following result. For a proof, see for example Proposition 17.24 in [Har92] or Theorem I.7.2 in [Har77].

Theorem 3.12. *(Projective dimension theorem) Let $X, Y \subset \mathbb{P}^n$ be varieties of pure dimensions k, l, respectively. Then, every irreducible component of $X \cap Y$ has dimension $\geq k+l-n$. Furthermore, if $k+l-n \geq 0$, then $X \cap Y$ is necessarily non-empty.*

Recall now the definition of the join from Talk 7.

Definition 3.13. Let $X, Y \subset \mathbb{P}^n$ be two varieties of pure dimensions k, l, respectively. Suppose there exist complementary linear subspaces $\mathbb{P}^m, \mathbb{P}^{n-m-1} \subset \mathbb{P}^n$ containing X and Y, respectively. The *join of X and Y* is defined to be the union

$$J(X, Y) = \bigcup_{x \in X, y \in Y} \overline{x, y}$$

Lemma 3.14. *Under the above assumptions,* $\dim(J(X, Y)) = k + l + 1$.

Furthermore, at a point $r \in \overline{p, q} \setminus \{p, q\}$, where $p \in X, q \in Y$ the tangent space $T_r J(X, Y)$ is spanned by $T_p X$ and $T_q Y$.

Proof. Propositions 11.37 and 16.14 in [Har92]. □

Lemma 3.15. *Under the above assumptions,* $\deg(J(X,Y)) = \deg(X)\cdot\deg(Y)$.

Proof. Consider a general $(m-k)$-plane $\Lambda' \subset \mathbb{P}^m$ and a general $(n-k-l-1)$-plane $\Lambda'' \subset \mathbb{P}^{n-m-1}$. Notice that both planes have the appropriate dimension to obtain the degrees of X and Y. That is, Λ' intersects X in $d = \deg(X)$ points $p_1, \dots, p_d$ and Λ'' intersects Y in $e = \deg(Y)$ points.

Consider the $(n-k-l)$-plane $\Lambda \subset \mathbb{P}^n$ spanned by Λ' and Λ''. Its intersection with $J(X,Y)$ will consist of the $d\cdot e$ lines $\overline{p_i, q_j}$ (here we are implicitly using the projective dimension theorem 3.12.

By Lemma 3.9, we may assume that Λ' and Λ'' intersect X and Y transversely, respectively. We claim that Λ and $J(X,Y)$ intersect generically transversely. Indeed, any point $r \in J(X,Y)$ lies in a line $\overline{p,q}$ with $p \in X$ and $q \in Y$. A general point $r \in J(X,Y)$ will not lie in either X or Y, so we may assume $r \neq p, q$. By Lemma 3.14, the join $J(X,Y)$ is smooth at such a point and

$$\begin{aligned} T_r\mathbb{P}^n &= T_p\mathbb{P}^m + T_q\mathbb{P}^{n-m-1} \\ &= T_pX + T_p\Lambda' + T_qY + T_q\Lambda'' \\ &= T_pX + T_qY + T_r\Lambda = T_rJ(X,Y) + T_r\Lambda. \end{aligned}$$

Hence, $J(X,Y)$ and Λ intersect generically transversely, and so by Lemma 3.11: $\deg(J(X,Y)) = \deg(J(X,Y)\cap\Lambda) = d\cdot e = \deg(X)\cdot\deg(Y)$. ☐

We are now ready to give a proof of Theorem 3.6.

Proof. (of Theorem 3.6) By Lemma 3.10, we may assume that $k+l=n$ and hence X and Y intersect transversely.

Consider the embeddings $i, j, k : \mathbb{P}^n \to \mathbb{P}^{2n+1}$ defined by

$$\begin{aligned} i &: [Z_0 : \dots : Z_n] \mapsto [Z_0 : \dots : Z_n : 0 : \dots : 0], \\ j &: [Z_0 : \dots : Z_n] \mapsto [0 : \dots : 0 : Z_0 : \dots : Z_n], \\ k &: [Z_0 : \dots : Z_n] \mapsto [Z_0 : \dots : Z_n : Z_0 : \dots : Z_n]. \end{aligned}$$

Set $\tilde{X} = i(X)$, $\tilde{Y} = j(Y)$, $J = J(\tilde{X}, \tilde{Y})$ and $L = k(\mathbb{P}^n)$. Notice that $L \cap J = \{[Z_0 : \dots : Z_n : Z_0 : \dots : Z_n] | [Z_0 : \dots : Z_n] \in X \cap Y\}$. Fix $r \in L \cap J$. Then, $r \in \overline{p,q}$ for $p \in \tilde{X}$ and $q \in \tilde{Y}$, but $r \neq p, q$ because L does not intersect $\tilde{X}$ nor $\tilde{Y}$. By Lemma 3.14, we have:

$$\begin{aligned} T_rJ + T_rL &= T_pi(X) + T_qj(Y) + T_rk(X) + T_rk(Y) \\ &= (T_pi(X) + T_rk(X)) + (T_qj(Y) + T_rk(Y)) \\ &= T_r\mathbb{P}^{2k+1} + T_r\mathbb{P}^{2l+1} = T_r\mathbb{P}^{2n+1} \end{aligned}$$

Hence, L and J intersect transversely. Finally, by Lemma 3.15, we have

$$\deg(X\cap Y) = \#(X\cap Y) = \#(L\cap J) = \deg(X)\cdot\deg(Y).$$

☐

3. **Stronger Bézout's Theorem.** In order to generalize Theorem 3.6, we will need to relax a bit the hypothesis of *generically transverse*. However, as Remark 3.7(v) exhibits, this will make us also change equation (1) by something weaker.

We begin by defining the *intersection multiplicity* $m_Z(X,Y)$ when X and Y are local complete intersections intersecting properly and Z is an irreducible component of $X \cap Y$ (for a more general treatment, see for example [Wei46]).

Definition 3.16. We do so in two steps:

(i) Suppose that $\dim(X) + \dim(Y) = n$. Then, every irreducible component Z of $X \cap Y$ is just a point p. We define $m_p(X,Y) := \dim_k (\mathcal{O}_{\mathbb{P}^n,p}/I(X) + I(Y))$. It should be mentioned that in this last equation, $I(X)$ actually denotes $I(X) \cdot \mathcal{O}_{\mathbb{P}^n,p}$.

(ii) Suppose X and Y have arbitrary dimension. A general $(n - \dim(Z))$-plane $\Gamma \subseteq \mathbb{P}^n$ intersects Z transversely and hence we may take $m_Z(X,Y) := m_p(X \cap \Gamma, Y \cap \Gamma)$ for $p \in Z \cap \Gamma$. Notice that $X \cap \Gamma$ and $Y \cap \Gamma$ have the right dimension so that $m_p(X \cap \Gamma, Y \cap \Gamma)$ corresponds to case (i).

Remark 3.17. Notice that this coincides with the definition of $I_p(C, C')$ for plane curves C, C' (cf. talk 11).

Proposition 3.18. *The intersection multiplicity has the following properties:*

(i) $m_Z(X,Y) \geq 1$.

(ii) $m_Z(X,Y) = 1$ *if, and only if,* X *and* Y *intersect transversely at a general point* $p \in Z$.

(iii) $m_Z(X \cup X', Y) = m_Z(X,Y) + m_Z(X',Y)$ *if* X *and* X' *have no common components.*

We are now ready to state the stronger version of Bézout's Theorem.

Theorem 3.19. *Let* $X, Y \subseteq \mathbb{P}^n$ *be varieties of pure dimension intersecting properly. Then*

$$\deg(X) \cdot \deg(Y) = \sum_Z m_Z(X,Y) \cdot \deg(Z), \tag{3}$$

where the sum ranges over all irreducible components Z *of* $X \cap Y$.

Remark 3.20. Notice that Proposition 1.1 can be obtained as a corollary of this.

4. **Some consequences of Bézout's Theorem.** We end this section by giving some direct applications of this theorem.

Corollary 3.21. *Let* $X \subseteq \mathbb{P}^n$ *be an irreducible variety. Then,* $\deg(X) = 1$ *if, and only if,* X *is linear.*

Proof. Exercise 2. □

Corollary 3.22. *Let* $X \subseteq \mathbb{P}^n$ *be an irreducible* k*-dimensional variety and* $\Omega \subseteq \mathbb{P}^n$ *be an* $(n-k)$*-plane. The following are equivalent:*

(i) Ω *gives the right degree of* X*, i.e. the intersection* $\Omega \cap X$ *consists of* $\deg(X)$ *points.*

(ii) X *and* Ω *intersect generically transversely.*

(iii) X *and* Ω *intersect transversely.*

Proof. "$(i) \Rightarrow (ii)$" : If Ω gives the right degree, then it intersects X at $d = \deg(X)$ points $p_1, \ldots, p_d$. In particular, Ω and X intersect properly, so we can apply equation (3) to obtain $\deg(X) = d = \sum_{i=1}^{d} m_{p_i}(X, \Omega)$. By Proposition 3.18(i), $m_{p_i}(X, \Omega) = 1$ for every i. By Proposition 3.18(ii), this implies that X and Ω intersect generically transversely.

"$(ii) \Rightarrow (i)$" : If X and Ω intersect generically transversely, then the weak version 3.6 already gives $\deg(X) = \deg(X \cap \Omega)$.

Finally, the equivalence $(ii) \Leftrightarrow (iii)$ was already established in Remark 3.2. □

Corollary 3.23. *Let $X, Y \subseteq \mathbb{P}^n$ be varieties of pure dimension intersecting properly. Then*

$$\deg(X \cap Y) \leq \deg(X) \cdot \deg(Y). \tag{4}$$

Proof. $\deg(X \cap Y) = \sum_Z \deg(Z) \leq \sum_Z m_Z(X,Y) \deg(Z) = \deg(X) \cdot \deg(Y)$. □

Corollary 3.24. *Let $X, Y \subseteq \mathbb{P}^n$ be varieties of pure dimension intersecting properly and satisfying*

$$\deg(X \cap Y) = \deg(X) \cdot \deg(Y). \tag{5}$$

Then, X and Y are smooth at a general point of any irreducible component of $X \cap Y$. In particular, if $\dim(X) + \dim(Y) = n$, then X and Y are smooth at all points of $X \cap Y$.

Proof. From the proof of Corollary 3.23, if equality (5) holds, then $m_Z(X,Y) = 1$ for every irreducible component Z of $X \cap Y$. By Proposition 3.18 this implies that $X \cap Y$ intersect generically transversely. In particular, they are smooth at a general point $p \in Z$ for every irreducible component Z. □

4. Varieties of minimal degree

We begin by describing curves of minimal degree and after that we jump to arbitrary dimensions.

1. **Curves of minimal degree.**

Definition 4.1. A variety $X \subset \mathbb{P}^n$ is called *nondegenerate* if there exists no hyperplane containing it.

Irreducible nondegenerate curves have a lower bound on their degree:

Proposition 4.2. *Let $C \subset \mathbb{P}^d$ be an irreducible nondegenerate curve. Then $\deg(C) \geq d$.*

Proof. Suppose $\deg(C) < d$. Pick d distinct points $p_1, \ldots, p_d \in C$. These span a hyperplane $H \subset \mathbb{P}^d$. If $\dim(C \cap H) = 0$, then C and H would intersect properly. By Corollary 3.23, this implies

$$d \leq \deg(C \cap H) \leq \deg(C) \cdot \deg(H) < d,$$

contradiction. Hence, we must have $\dim(C \cap H) = 1$. But this together with the facts that $C \cap H \subseteq C$ and C irreducible implies $C \cap H = C$ and thus $C \subseteq H$, contradicting the nondegeneracy of C. □

As a consequence, it makes sense to talk about irreducible nondegenerate curves of minimal degree d. We will show that these correspond exactly to rational normal curves. We recall (again) their definition.

Definition 4.3. A *rational normal curve* $C \subset \mathbb{P}^d$ is a variety which is projectively equivalent to the image of the map

$$\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$$
$$[X_0 : X_1] \mapsto [X_0^d : X_0^{d-1}X_1 : \ldots : X_1^d].$$

Proposition 4.4. *Rational normal curves $C \subset \mathbb{P}^d$ are exactly the irreducible nondegenerate curves of minimal degree, that is, of degree d.*

Proof. Rational normal curves have minimal degree d by Remark 2.13. They are also irreducible and nondegenerate (cf. Proposition 7.2 in Talk 2).

Conversely, suppose $C \subset \mathbb{P}^n$ is an irreducible, nondegenerate curve of degree d. Pick $p_1, \dots, p_{d+1} \in C$ distinct points. By Corollary 3.23, these are linearly independent. Since $p_1, \dots p_d$ span a hyperplane that intersects C at exactly d points, by Corollary 3.24, C is smooth at these points (in fact, since these have been chosen arbitrarily, C is smooth everywhere). Hence, the tangent space is well-defined at each point of C.

Consider the $(d-2)$-plane Λ spanned by $p_1, \dots, p_{d-1}$. We define a regular map

$$\varphi : \Lambda^* \cong \mathbb{P}^1 \to C \subset \mathbb{P}^d$$

by sending a hyperplane $H \supset \Lambda$ to the point in $H \cap C$ that is none of $p_1, \dots, p_{d-1}$, or to p_i if H is tangent to C at p_i. This map has a regular inverse ψ that sends a point $p \in C \smallsetminus \{p_1, \dots, p_{d-1}\}$ to the hyperplane $H \in \Lambda^*$ spanned by $p_1, \dots, p_{d-1}, p$ and sends p_i to the hyperplane spanned by $p_1, \dots, p_{d-1}$ and the line tangent to C at p_i.

Hence, we have a regular isomorphism

$$\varphi : \mathbb{P}^1 \xrightarrow{\sim} C \subset \mathbb{P}^d.$$

By performing a projective equivalence, we can assume $p_1 = [1 : 0 : \dots : 0], p_2 = [0 : 1 : \dots : 0], \dots p_{d+1} = [0 : 0 : \dots : 1]$. Since φ is defined on $\mathbb{P}^1$, it must be given by homogeneous polynomials $f_0(X_0, X_1), \dots, f_d(X_0, X_1)$ of the same degree, namely

$$\varphi([X_0 : X_1]) = [f_0(X_0, X_1), \dots, f_d(X_0, X_1)].$$

By intersecting C with the hyperplanes spanned by $p_1, \dots, \hat{p}_i, \dots p_{d+1}$, we obtain that each f_i has d distinct roots. Since C has degree d it cannot happen that they have more than d roots, as this would contradict Bézout's Theorem. Hence, the polynomials f_i have degree d.

By the characterization of rational normal curves given in Talk 2, it suffices to check that the polynomials f_i are linearly independent. Suppose $\sum_i a_i f_i(X_0, X_1) = 0$. Substituting (X_0, X_1) by the points $\varphi^{-1}(p_i)$ yields $0 = a_i f_i(\varphi^{-1}(p_i)) = a_i$. □

2. **Varieties of minimal degree of arbitrary dimension.** The classification of irreducible nondegenerate curves of minimal degree is thus particularly simple. One might ask if a similar classification exists for varieties of higher dimensions. The answer is affirmative, but it is not as simple as for curves. In order to state it, we first show what the minimal degree is in terms of the dimension of the variety and of the ambient space.

Proposition 4.5. *Let $X \subset \mathbb{P}^n$ be an irreducible nondegenerate k-dimensional variety. Then,* $\deg(X) \geq n - k + 1$.

Proof. (Idea) By induction on k. The case $k = 1$ is Proposition 4.2. Suppose that $k > 1$ and that the result is true for dimensions smaller than k. The intersection of X with a general hyperplane $H \subset \mathbb{P}^n$ is an irreducible nondegenerate $(k-1)$-dimensional subvariety of $H \cong \mathbb{P}^{n-1}$ (this is not easy, cf. Proposition 18.10 in [Har92]). Then:

$$\deg(X) \overset{(1)}{=} \deg(X \cap H) \overset{(2)}{\geq} (n-1) - (k-1) + 1 = n - k + 1,$$

where (1) follows from Remark 2.15 and (2) is the induction hypothesis. □

Still before stating the classification theorem, we define (or recall) some important varieties.

The following two definitions where already introduced in Talks 5 and 3, respectively.

Definition 4.6. Let $X \subset \mathbb{P}^n$ be a variety and $p \notin X$ be a point. The *cone of X with vertex p* is the join $J(X, p)$.

Definition 4.7. A *Veronese variety of degree d and dimension n* is a variety which is projectively equivalent to the image of the map

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^N$$
$$[X_0 : \ldots : X_n] \mapsto [\ldots X^I \ldots],$$

where X^I ranges over all degree d monomials in $X_0, \ldots, X_n$, and $N = \binom{n+d}{d} - 1$.

We will be particularly interested in the case $d = n = 2$: a *Veronese surface* is a variety which is projectively equivalent to the image of the map

$$\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$$
$$[X_0 : X_1 : X_2] \mapsto [X_0^2 : X_1^2 : X_2^2 : X_0X_1 : X_0X_2 : X_1X_2].$$

Definition 4.8. A *rational normal scroll $S_{k,l} \subset \mathbb{P}^n$* (with $n = k+l+1$) is a variety constructed as follows:

(1) Pick two complementary linear subspaces Λ, Λ' of dimensions k, l, respectively.
(2) Pick rational normal curves $C \subset \Lambda$, $C' \subset \Lambda'$.
(3) Pick an isomorphism $\varphi : C' \to C$.
(4) Set $S_{k,l} = \bigcup_{p \in C'} \overline{p, \varphi(p)}$.

Remark 4.9. The projective isomorphism class of $S_{k,l}$ does not depend on any choice.

We are finally ready to state the classification theorem of varieties of minimal degree. Proofs can be found in [Har81], [Xam81], [EiHa87].

Theorem 4.10. *(Del Pezzo - Bertini) Let $X \subset \mathbb{P}^n$ be an irreducible nondegenerate k-dimensional variety of minimal degree $n - k + 1$. Then X is exactly one of the following:*

(i) a quadric hypersurface of rank at least 5,
(ii) a cone over the Verones surface, or
(iii) a rational normal scroll.

5. Exercises

1. Let $X, Y \subset \mathbb{P}^n$ be varieties of pure dimension intersecting generically transversely, and $X \cap Y \neq \varnothing$. Then, X and Y intersect properly.

Solution. Fix an irreducible component Z of $X \cap Y$. We want to see that it has dimension $\dim(X) + \dim(Y) - n$. At a general point $p \in Z$ (hence at some smooth point of X and Y), we have

$$\dim(Z) \leq \dim(T_pZ) \leq \dim(T_pX \cap T_pY) = \dim(X) + \dim(Y) - n.$$

The other inequality follows immediately from the projective dimension theorem 3.12.

2. Let $X \subset \mathbb{P}^n$ be an irreducible variety. Prove that $\deg(X) = 1$ if, and only if, X is linear.

Solution. "$\Rightarrow$" is Example 2.12. For the converse, we proceed by induction on $k = \dim(X)$. If $k = 0$, then X is just a point, so it is linear. If $k > 0$, then we can pick $k+1$ linearly independent points $p_0, \ldots, p_k \in X$. Set $L = \operatorname{span}(p_0, \ldots, p_k)$ and let $H \subset \mathbb{P}^n$ be a hyperplane containing L.

We claim that $X \subseteq H$. Indeed, suppose $X \not\subseteq H$. Then X and H intersect properly (by the projective dimension theorem). Now, by Bézout's Theorem 3.19 we have

$$1 = \deg(X) \cdot \deg(H) = \sum_Z m_Z(X, Y) \deg(Z).$$

Since $m_Z(X, Y), \deg(Z) \geq 1$ for every irreducible component Z of $X \cap H$, this intersection must be irreducible. Hence, $\deg(X \cap H) = 1$ and $\dim(X \cap H) = k - 1$. By induction, $X \cap H \subset \mathbb{P}^n$ is a $(k-1)$-plane that contains $p_0, \ldots, p_k$, i.e. k linearly independent points, contradiction.

To finish the the problem, just notice that

$$X \subseteq \bigcap_{H \supseteq L \text{ hyperplane}} H = L.$$

Since L is irreducible and $\dim(X) = \dim(L) = k$, the equality $X = L$ holds.

3. Show that the degree of the Veronese variety $\nu_d(\mathbb{P}^n)$ equals d^n. Deduce that the Veronese surface and its cone have minimal degree.

Solution. Write $X = \nu_d(\mathbb{P}^n)$. By Remark 2.15, $\deg(X) = \deg(X \cap H_1 \cap \ldots \cap H_n)$, where H_i are general hyperplanes in $\mathbb{P}^N$ (recall that $N = \binom{n+d}{d} - 1$). Since $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ is an isomorphism onto its image, the intersection $X \cap H_1 \cap \ldots \cap H_n$ is in bijective correspondence with $\nu_d^{-1}(H_1) \cap \ldots \cap \nu_d^{-1}(H_n)$.

Hence, $\deg(X) = \#(\nu_d^{-1}(H_1) \cap \ldots \cap \nu_d^{-1}(H_n))$. Now, we claim that each $\nu_d^{-1}(H_i)$ is a general hypersurface of $\mathbb{P}^n$ of degree d. Indeed, if H_i is the zero locus of $a_0 Z_0 + \ldots + a_N Z_N$, then $\nu_d^{-1}(H_i)$ is the zero locus of $\sum a_I X^I$. Since this sum is over all multi-indices I whose sum equals d, and the coefficients a_I are general, the claim follows.

The final trick that does the job consists of using a slight generalization of Lemma 3.9. Namely, a general hypersurface intersects a variety transversely (the proof of Lemma 3.9 can be generalized to prove this statement). Hence, the hypersurfaces $\nu_d^{-1}(H_1), \ldots, \nu_d^{-1}(H_n)$ intersect transversely and by Bézout's Theorem, we have

$$\begin{aligned}\deg(X) &= \#(\nu_d^{-1}(H_1) \cap \ldots \cap \nu_d^{-1}(H_n)) = \deg(\nu_d^{-1}(H_1) \cap \ldots \cap \nu_d^{-1}(H_n)) \\ &= \deg(\nu_d^{-1}(H_1)) \cdots \deg(\nu_d^{-1}(H_n)) = d^n.\end{aligned}$$

For the second part, set $d = n = 2$. Then $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ and $\deg(\nu_2(\mathbb{P}^2)) = 4 = 5 - 2 + 1$ so the Veronese surface has minimal degree.

For the cone over the Veronese surface, notice that taking a general hyperplane section of it, yields a variety that is projectively equivalent to $\nu_2(\mathbb{P}^2)$, but now living in the ambient space $\mathbb{P}^6$. In particular, taking the cone does not change the degree. Hence, $\deg(\text{cone over } \nu_2(\mathbb{P}^2)) = 4 = 6 - 3 + 1$, as wanted.

References

[Har77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 52, Springer, New York, 1977.

[Har92] J. Harris, *Algebraic Geometry. A First Course*, Graduate Texts in Mathematics 133. Springer 1992.

[Wei46] A. Weil. *Foundations of Algebraic Geometry*, Amer. Math. Soc., Colloquium Publ. 29 (1946) (revised and enlarged edition 1962)

[Xam81] S. Xambò. *On projective varieties of minimal degree*, Collect. Math. **32** (1981), 149-163.

[EiHa87] D. Eisenbud, J. Harris, *On varieties of minimal degree (a centennial account)*, Bowdoin, 1985 (Brunswick, Maine, 1985). Proceedings of Symposia in Pure Mathematics. **46** (1987), no.1, 3-13.

[Har81] J. Harris, *A bound on the geometric genus of projective varieties*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **8** (1981), 35-68.