EXOTIC EMBEDDINGS

ALEJANDRO GARCÍA RIVAS

1. Introduction

In this talk, I will discuss some of what we know about exotic embeddings of surfaces in 4-manifolds. We start by defining what we mean by "exotic embedding".

Definition 1.1. A pair of surfaces in a smooth 4-manifold X are said to be *exotically knotted* if they are isotopic through homeomorphisms but not through diffeomorphisms of X.

Some rigidity statements are already known for quite a while:

Theorem 1.2 (Abhyankar-Moh, 1975)**.** *All polynomial embeddings $\mathbb{C} \to \mathbb{C}^2$ are algebraically equivalent. In particular, there are no exotically knotted $\mathbb{C}$ algebraically embedded in $\mathbb{C}^2$.*

Theorem 1.3 (Fintushel-Stern, 1999)**.** *Let X^4 be a closed, simply-connected Kähler surface. Any two smooth complex curves in X^4 which are homologous are also smoothly isotopic.*

On the other hand, there are some known results about existence of exotically knotted surfaces.

Theorem 1.4 (Freedman, 1985)**.** *There are exotically knotted $\mathbb{R}^2$ propperly embedded in $\mathbb{R}^4$.*

Theorem 1.5 (Finashin-Kreck-Viro, 1988)**.** *There are exotically knotted closed nonorientable surfaces in $\mathbb{R}^4$.*

Remark 1.6*.* The orientable case is still open!

The following three results are the ones we will be concerned about in this talk.

Theorem 1.7 (Hayden, 2020)**.** *There exist pairs of exotically knotted disks properly embedded in B^4.*[1]

Theorem 1.8 (Hayden, 2021)**.** *There exist pairs of exotically knotted orientable surfaces in B^4 of arbitrary number of punctures and genus.*

Theorem 1.9. *For every $n \in \mathbb{N}$ there is a 2^n-tuple of pairwise exotically knotted disks in B^4 relative boundary.*

2. A pair of exotically knotted disks in B^4

In this section, we prove Theorem 1.7. Consider the following 3-component link:

[1] Every embedding will be assumed to be proper from now on.

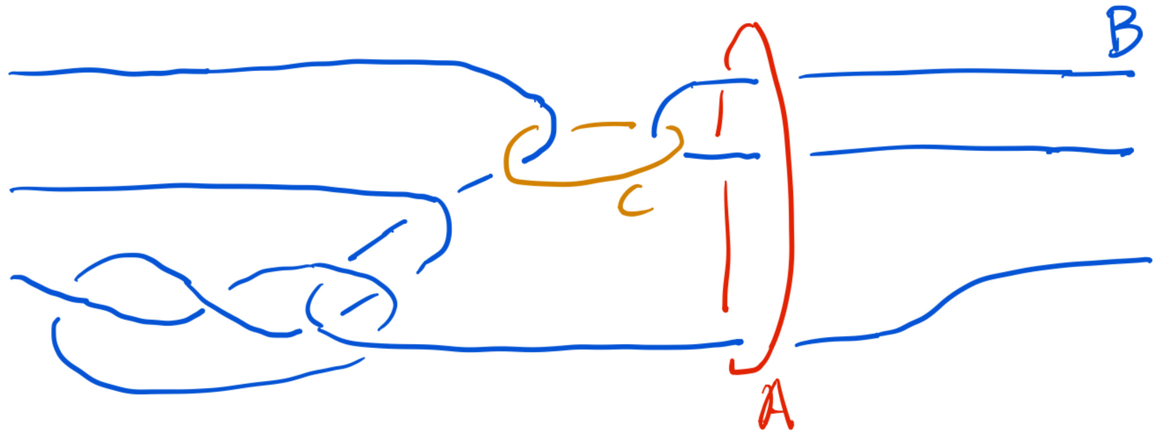

FIGURE 1. The 3-component link $A \cup B \cup C$. The knot B is supposed to be closed in the obvious way (just as when taking the closure of a braid).

Note that $A \cup C, B \cup C = U^2$ are the unlink, while $A \cup B$ is a Hopf link. Now, consider the 4-manifolds B_1 and B_2 given by the Kirby diagrams

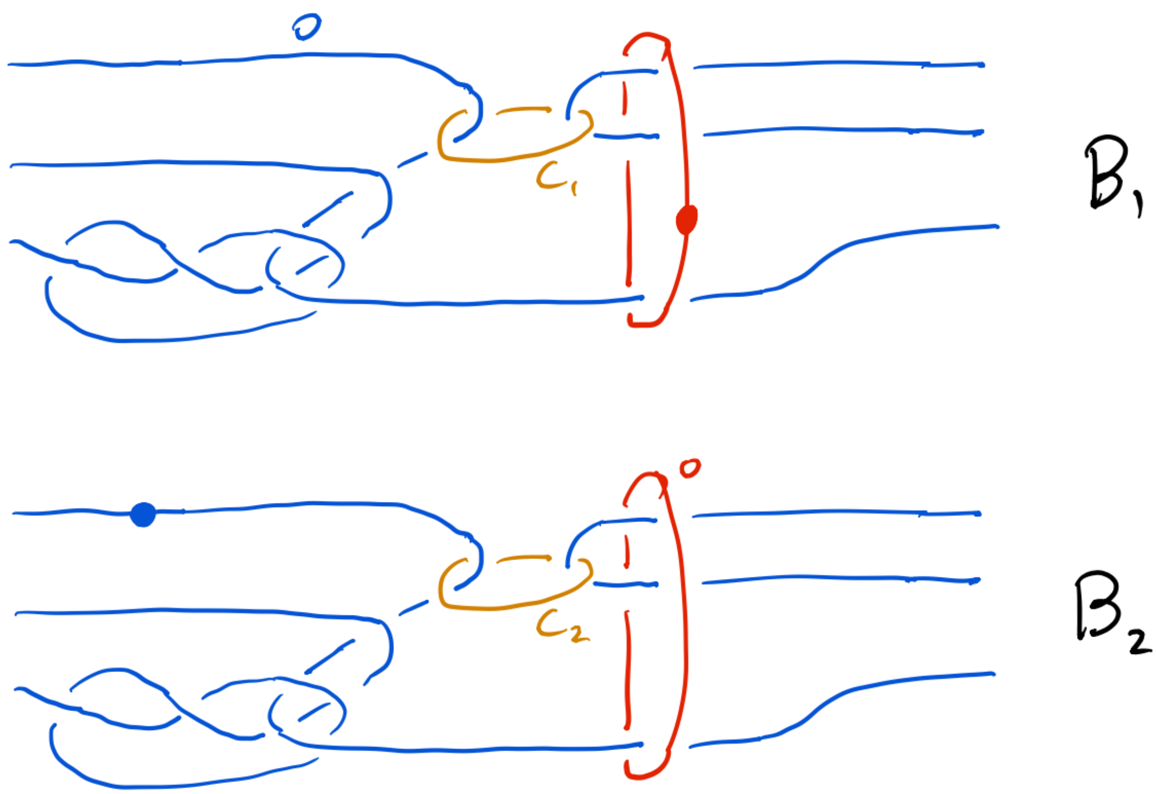

FIGURE 2. The 4-manifolds B_1 and B_2 together with the knots C_1 and C_2.

Notice that both B_1 and B_2 are 4-balls. Let $\varphi : \partial B_1 \cong S^3 \to S^3 \cong \partial B_2$ be the diffeomorphism that swaps the dotted circle and the 0-framed 2-handle. Note that $\varphi(C_1) = C_2$. Hence, C_1 and C_2 define the same knot K in S^3.

How does K look in S^3? We start with the Kirby diagram for B_1 above and slide the knot C_1 twice along the 2-handle, so that we can cancel the 1- and 2-handles while leaving the knot unaffected. This is done in Figures 3-6

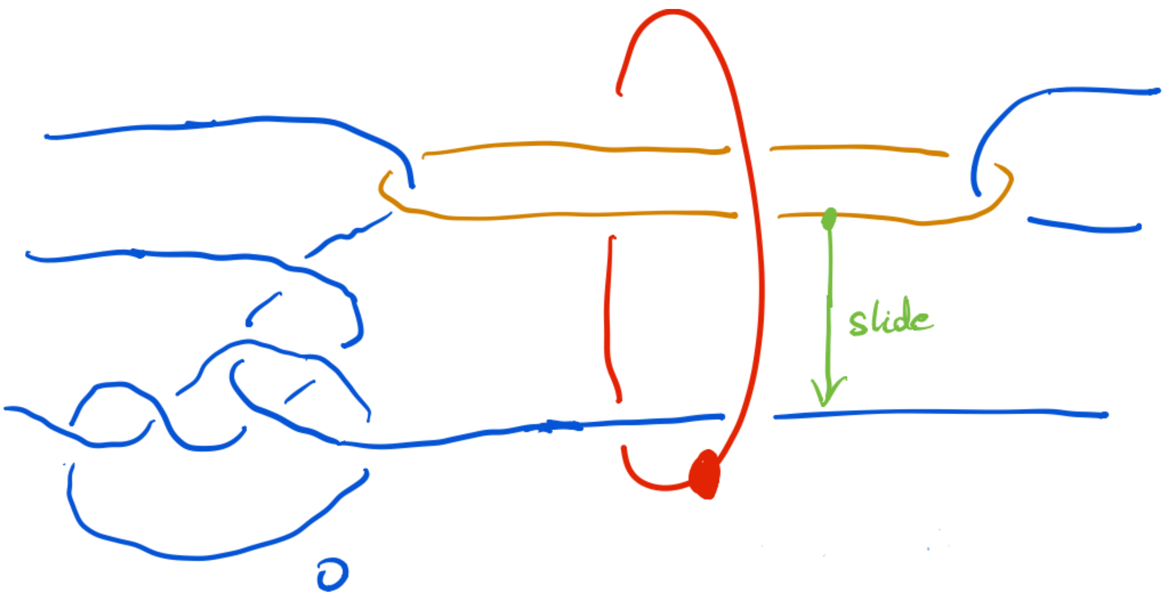

FIGURE 3. B_1 before the first slide.

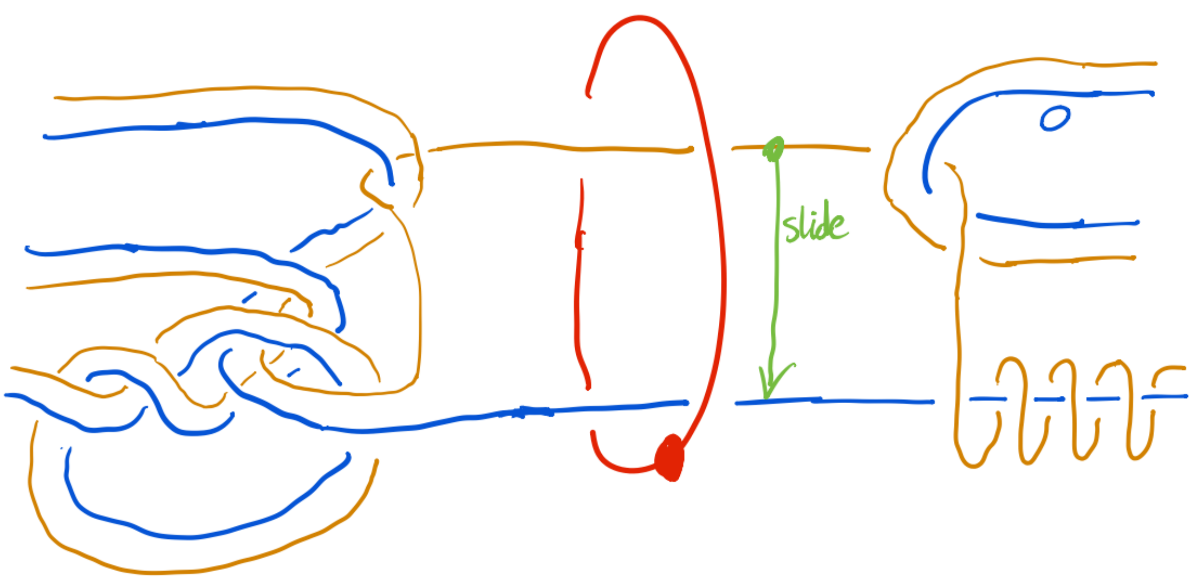

FIGURE 4. B_1 before the second slide.

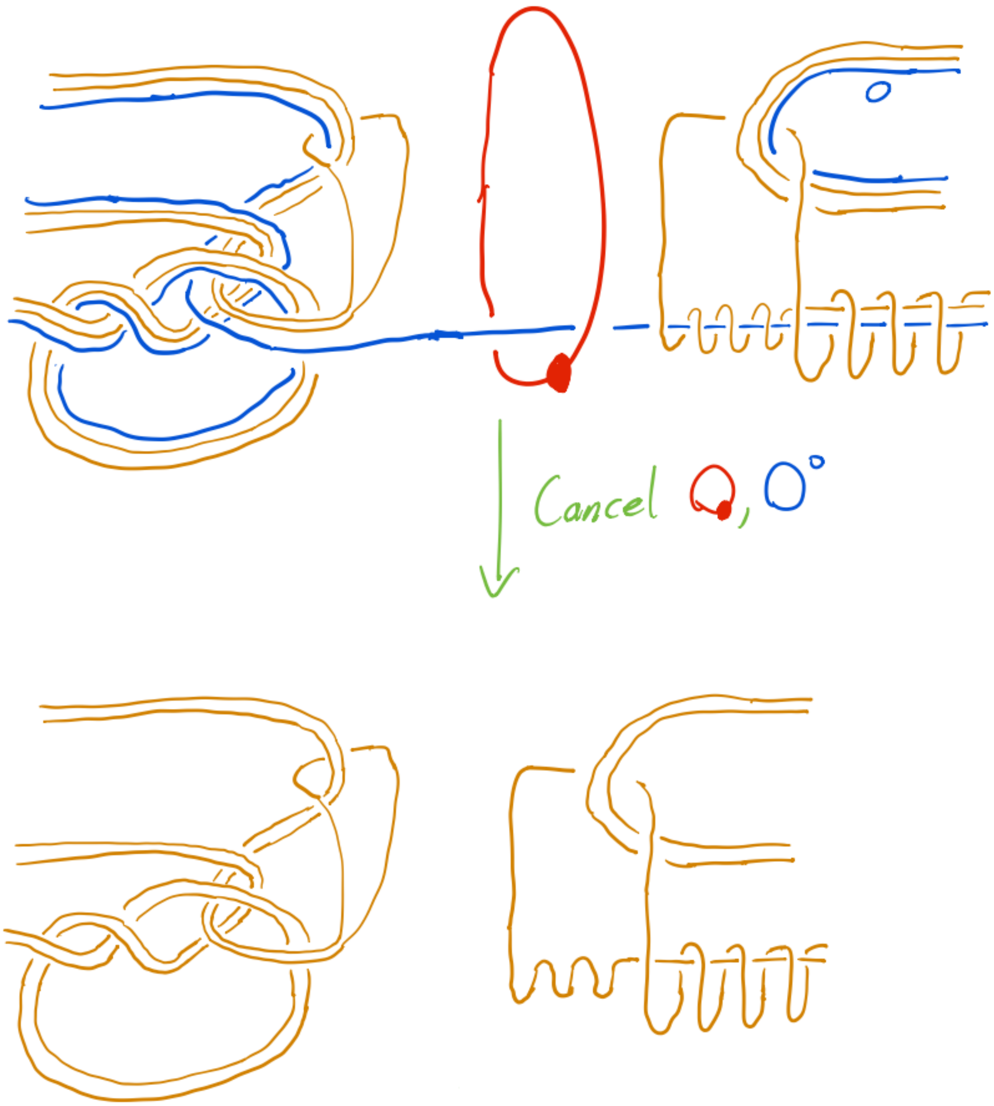

FIGURE 5. B_1 after the slides and after cancelling the 1- and 2-handles.

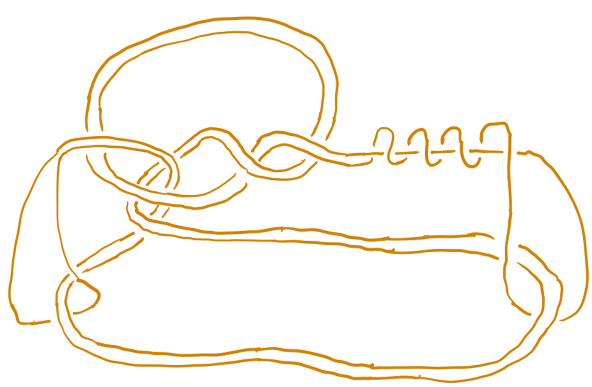

FIGURE 6. The knot K in S^3.

Note also that C_1 and C_2 bound slice disks D_1 and D_2, respectively:

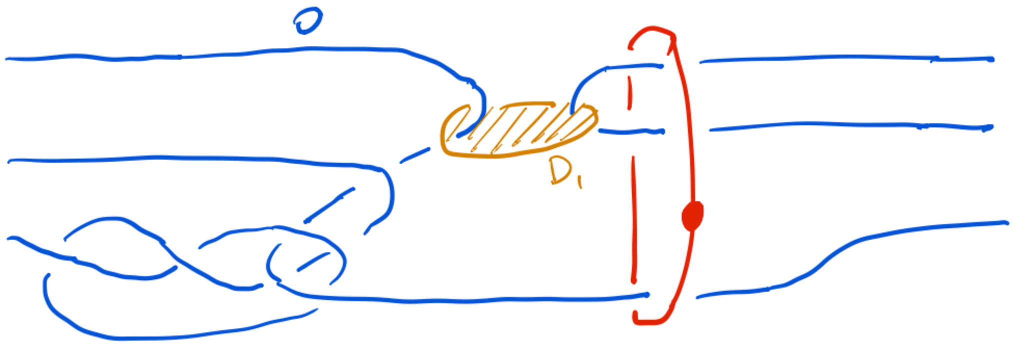

FIGURE 7. The slice disk D_1.

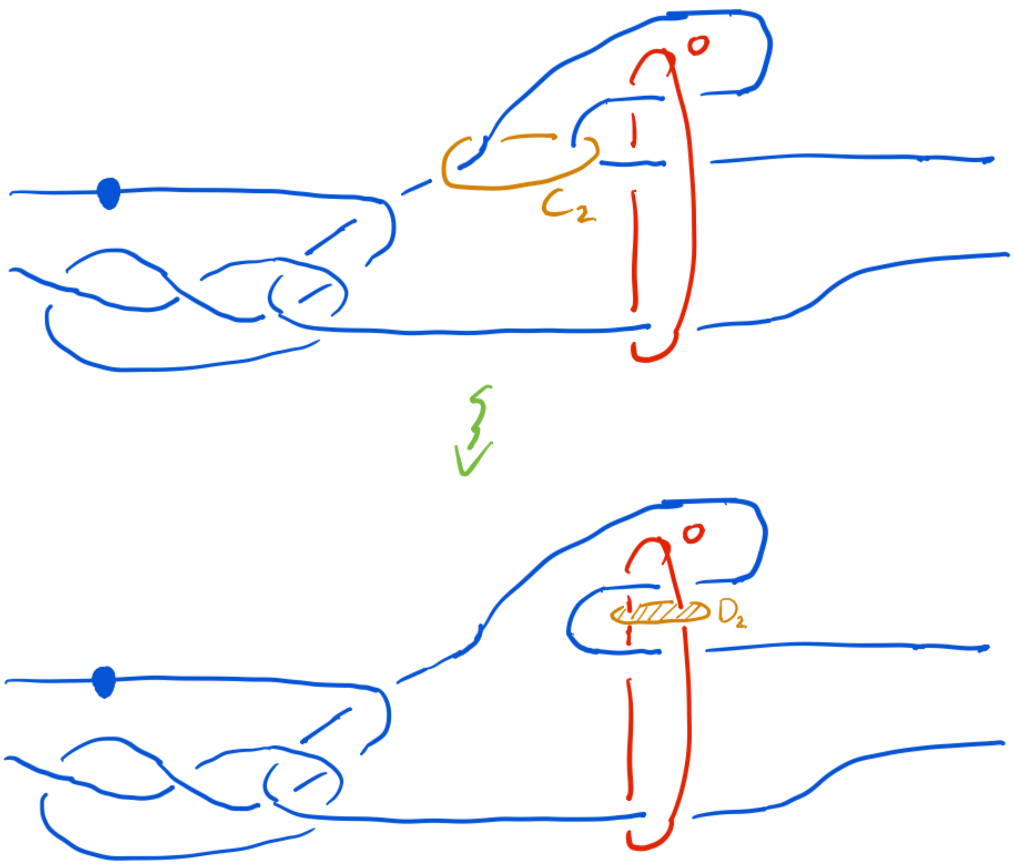

FIGURE 8. The slice disk D_2.

Our goal is to show that these disks are exotically knotted. We will think of these disks as living in the standard B^4. Without loss of generality, we can assume that they have the same boundary K in S^3.

Claim 2.1. $D_1 \simeq^{\partial}_{top} D_2$

This will follow from the following theorem:

Theorem 2.2 (Conway-Powell, 2019)**.** *If* $D_1, D_2 \overset{top}{\hookrightarrow} B^4$*,* $\partial D_1 = \partial D_2$*,* $\pi_1(B^4 \setminus D_1) \cong \pi_1(B^4 \setminus D_2) \cong \mathbb{Z}$*, then* $D_1 \simeq^{\partial}_{top} D_2$*.*

Proof. (of Claim 2.1) By the theorem, it suffices to prove $\pi_1(B^4 \setminus D_1) \cong \pi_1(B^4 \setminus D_2) \cong \mathbb{Z}$. Let's start with $B^4 \setminus D_1$. We have a chain of homotopy equivalences:

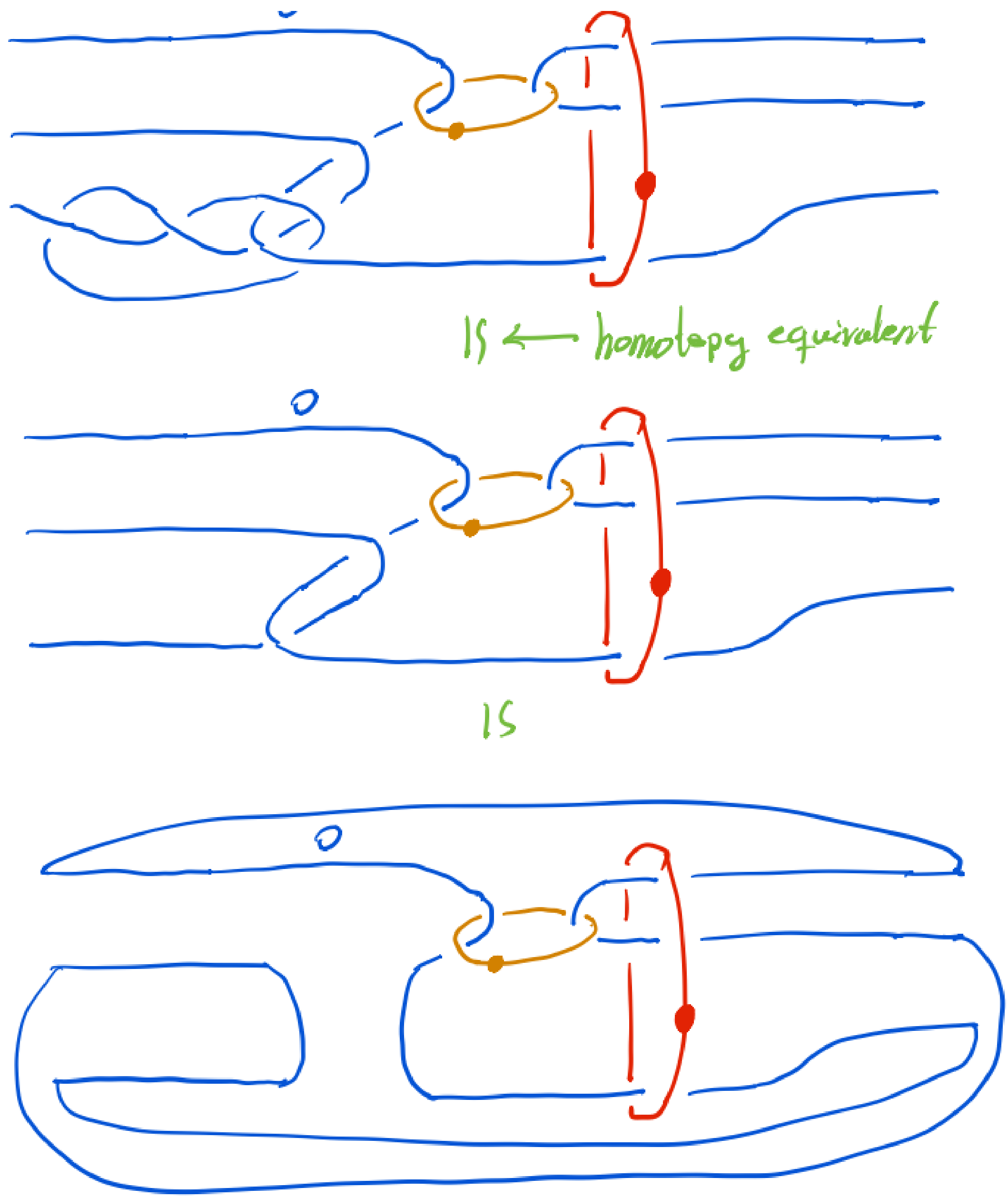

FIGURE 9. Chain of homotopy equivalences.

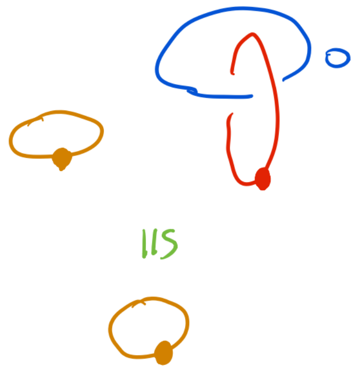

FIGURE 10. Handle cancellation.

Hence, $\pi_1(B^4 \setminus D_1) \cong \pi_1(\text{1-handle}) \cong \mathbb{Z}$. Now, let's work with $B^4 \setminus D_2$. By the isotopies of the appendix, $B^4 \setminus D_2$ has Kirby diagram:

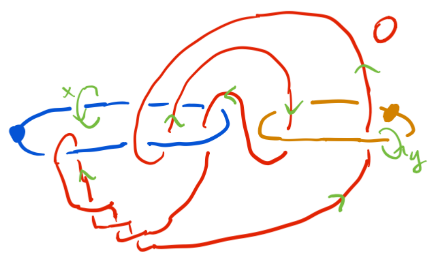

FIGURE 11. Kirby diagram of $B^4 \setminus D_2$ together with generators x, y of its fundamental group.

We simply compute:

$$\pi_1(B^4 \setminus D_2) \cong \langle x, y | y^{-1}x^{-1}yxx^{-1} \rangle \cong \langle x, y | y^{-1}x^{-1}y = e \rangle \cong \langle x, y | x^{-1}y = y \rangle = \langle x, y | x = e \rangle \cong \mathbb{Z}.$$

□

Now, let's prove that D_1 and D_2 are not smoothly isotopic. We will argue by contradiction, so suppose $D_1 \simeq_{sm} D_2$.

Claim 2.3. *There exists a diffeomorphism $F : E(D_1) \to E(D_2)$ between the exteriors of the disks, such that $F(S^3 \setminus \nu K) = S^3 \setminus \nu K$.*

Proof. Since $D_1 \simeq_{sm} D_2$, there is a diffeomorphism $G : (B^4, D_1) \cong (B^4, D_2)$. Since G takes fixes K setwise, we can isotope it to obtain the desired F. □

Claim 2.4. *$F_{|S^3 \setminus \nu K}$ is smoothly isotopic to $id_{|S^3 \setminus \nu K}$.*

In order to prove this, we will need some statements about hyperbolic knots.

Definition 2.5. A knot K is *hyperbolic* if its complement admits a complete Riemannian metric of constant negative curvature.

Example 2.6. The following knots are hyperbolic:

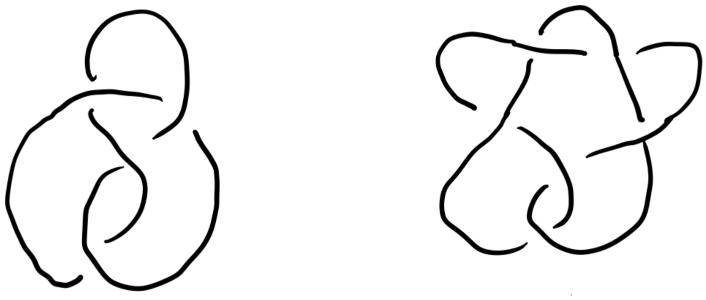

FIGURE 12. The figure eight knot and the Stevedore knot, respectively.

Theorem 2.7 (Hatcher, 1976)**.** *If K is hyperbolic, then $\pi_0 Diff(S^3 \setminus K) \cong \pi_0 Isom(S^3 \setminus K)$, i.e. any self-diffeomorphism of $S^3 \setminus K$ is isotopic to an isometry.*

Proof. (of Claim 2.4) Using SnapPy and Sage, one gets that K is hyperbolic and has trivial isometry group. By the theorem above, any diffeomorphism $S^3 \setminus K \to S^3 \setminus K$ is isotopic to $id_{S^3 \setminus K}$. □

By Claim 2.4, we can isotope $F : E(D_1) \to E(D_2)$ so that $F_{|S^3 \setminus \nu K} = id_{S^3 \setminus \nu K}$ at the expense of making the tubular neighborhood νK larger. Hence, on the boundary, F looks like:

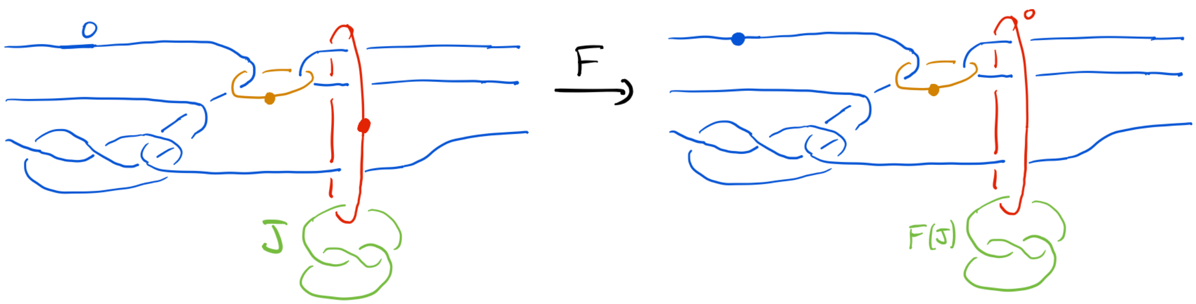

FIGURE 13. The diffeomorphism $F : E(D_1) \to E(D_2)$ restricted to the boundary. Notice, in particular, where F maps J.

Notice that $F(J)$ bounds a smoothly embedded punctured torus in $E(D_2)$, namely its Seifert surface with interior pushed inside the interior of $E(D_2)$. Since $F : E(D_1) \to E(D_2)$ is a diffeomorphism, J will also bound a smoothly embedded punctured torus in $E(D_2)$. Hence, in order to reach a contradiction it will suffice to show that this is not the case.

Recall from last talk:

Theorem 2.8 (Lisca-Matic, 1998)**.** *Let W^4 be a Stein domain with $M^3 = \partial W$. Let J be a Legendrian knot in M which boundss an orientable surface Σ. Then*

$$tb(J) + |rot(J)| \leq 2g(\Sigma) - 1.$$

First notice that $E(D_1)$ admits the structure of a Stein domain:

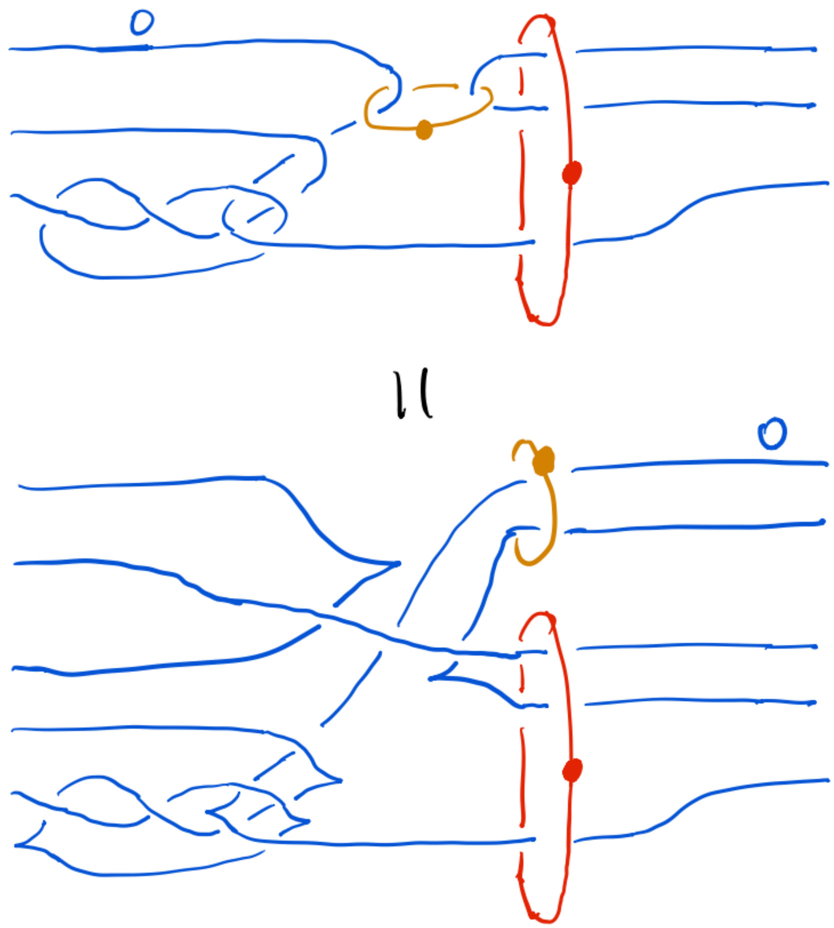

FIGURE 14. A Legendrian Kirby diagram for $E(D_1)$ in standard form.

Indeed, the blue curve has $tb = writhe - \#right\ cusps = 4 - 3 = 1$. Since the only 2-handle has framing $tb-1$, $E(D_1)$ admits a Stein structure. Now, the knot J can be pictured as

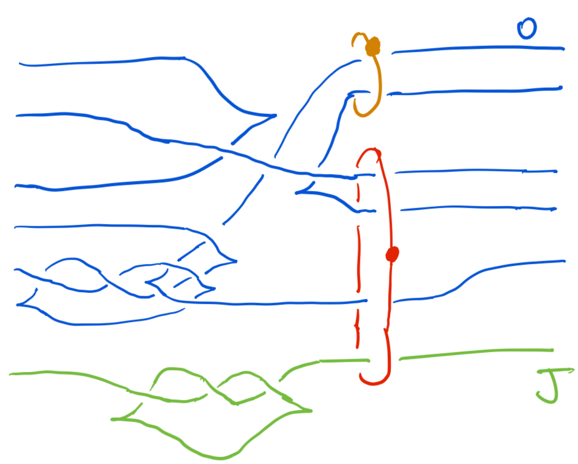

FIGURE 15. The knot J in $E(D_1)$.

and has $tb(J) = 3, |rot(J)| = 0$. But using Theorem 2.8, we have $3 \leq 2g(\Sigma) - 1$, so $g(\Sigma) \geq 2$, i.e. J does not bound a smoothly embedded punctured torus. This is the contradiction we were looking for, so we have proven Theorem 1.7.

3. Exotically knotted surfaces in B^4

In this section we prove Theorem 1.8 and Theorem 1.9. Consider the knot K and the ribbon disks D and D'.

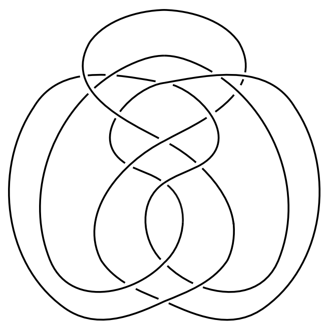

Figure 16. The knot K.

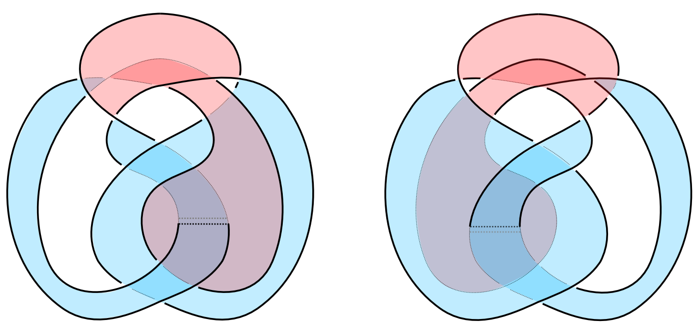

Figure 17. The disks D and D'.

Claim 3.1. $D \simeq^{\partial}_{top} D'$.

Proof. Again, by Theorem 2.2, it suffices to prove $\pi_1(B^4 \setminus D^2) \cong \mathbb{Z}$. Following the algorithm to construct Kirby diagrams of complements of surfaces, we obtain a Kirby diagram for $B^4 \setminus D^2$:

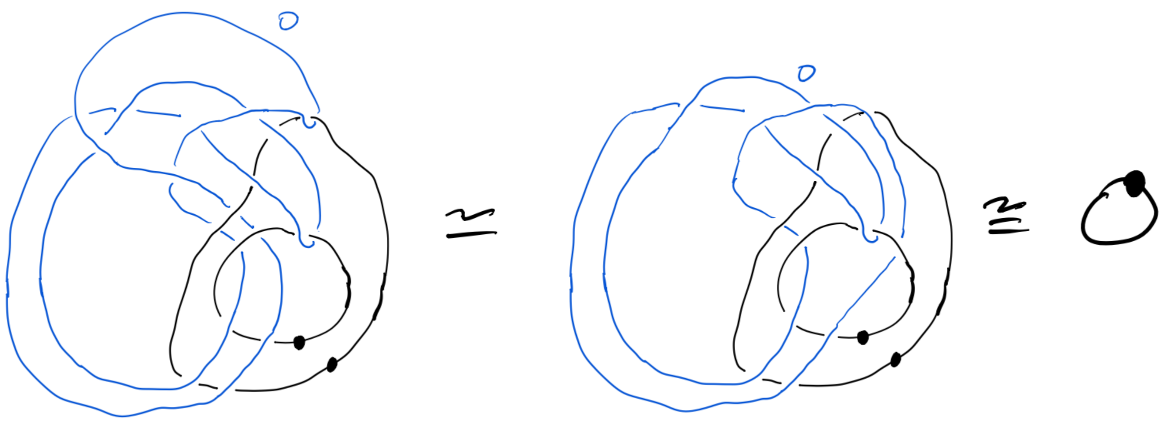

FIGURE 18. Chain of homotopy equivalences of $B^4 \setminus D^2$.

Since π_1(1-handle) $\cong \mathbb{Z}$, we have $\pi_1(B^4 \setminus D^2) \cong \mathbb{Z}$. Since there is an obvious diffeomorphism $(B^4, D) \cong (B^4, D')$, we also have $\pi_1(B^4 \setminus D') \cong \mathbb{Z}$. □

Now, consider the annuli

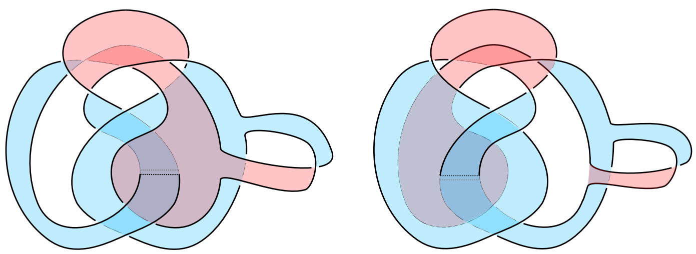

FIGURE 19. The annuli A and A'.

Since D and D' were topologically isotopic relative boundary, so will A and A'. We will show, though, that $A \not\simeq_{sm} A'$.

Claim 3.2. *The double branched cover $\Sigma_2(B^4, A')$ contains a smoothly embedded 2-sphere S with $[S]^2 = -2$.*

Proof. A' contains a Hopf annulus. The obvious disk in the picture below lifts to a sphere of square -2 in the double branched cover $\Sigma_2(B^4, A')$.

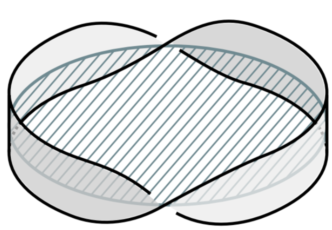

FIGURE 20. Hopf annulus located in A'.

□

Claim 3.3. *$\Sigma_2(B^4, A)$ does not contains a smoothly embedded 2-sphere S with $[S]^2 = -2$.*

Notice that this will prove $A \not\cong_{sm} A'$.
Recall from last talk:

Theorem 3.4 (Lisca-Matic, 1998)**.** *If S is a smoothly embedded surface in a Stein domain W^4 such that $[S] \neq 0$ in $H_2(W)$, then:*

$$[S]^2 + |\langle c_1(W), [S]\rangle| \leq 2g(S) - 2.$$

For every 2-handle h_i attached along a Legendrian knot K_i, we have $\langle c_1(W), h_i\rangle = rot(K_i)$.

Proof. (of Claim 3.3) After some long computations, a Kirby diagram for $\Sigma_2(B^2, A)$ is

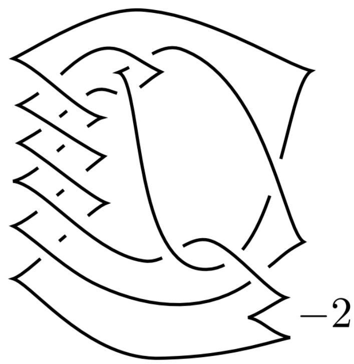

FIGURE 21. Kirby diagram for $\Sigma_2(B^4, A)$ in standard form.

Let J denote the knot of the above figure. One can easily compute $tb(J) = -1$ and $rot(J) = -2$. Since the framing of the 2-handle is -2, it follows that $\Sigma_2(B^4, A)$ admits a Stein structure. Now, suppose S is a smoothly embedded 2-sphere with $[S]^2 = -2$. In particular, $[S] =\neq 0$ is some multiple of the homology class h represented by the 2-handle. By Theorem 3.4, we have $\langle c_1(\Sigma_2(B^4, A), [S]\rangle \neq 0$. Hence, also:

$$2g(S) - 2 \geq [S]^2 + |\langle c_1(\Sigma_2(B^4, A), [S]\rangle| > -2.$$

It follows that $g(S) > 0$, contradiction. □

A similar argument works for the tori T and T'

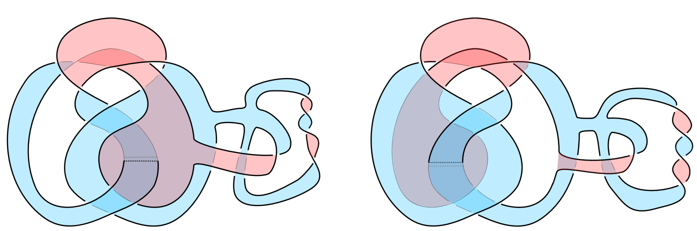

FIGURE 22. The tori T and T'.

Now, to obtain exotically knotted surfaces of arbitrary number of punctures and genus, we use the following facts.

Fact 3.5. *Any orientable surface with boundary is a boundary connected sum of tori and annuli.*

Fact 3.6. *Double branched covers behave well under boundary connected sums. Namely:*
$$\Sigma_2(B^4, F_1 \natural F_2) \cong \Sigma_2(B^4, F_1) \natural \Sigma_2(B^4, F_2).$$

Consider the annulus A_0 and the torus T_0.

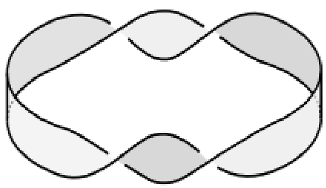

FIGURE 23. The annulus A_0.

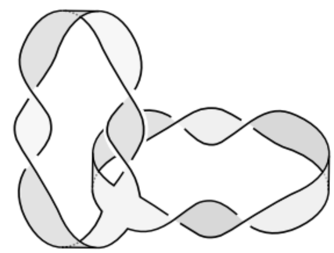

FIGURE 24. The torus T_0.

We are now ready to sketch a proof of Theorem 1.8

Proof. (of Theorem 1.8) For genus equal to 0, pick $F = A \natural_m A_0$ and $F = A' \natural_m A_0$. By the facts above, the same argument as for A and A' will carry through, as $\Sigma_2(B^4, A_0)$ has classes of square at most -4, i.e. boundary summing with A_0 will not make a sphere of square -2 appear. For positive genus, pick $F = T \natural_m A_0 \natural_k T_0$ and $F = T' \natural_m A_0 \natural_k T_0$ and the same idea applies. □

Along the way, we have also proven

Corollary 3.7. *All the pairs of 4-manifolds $\Sigma_2(B^4, F)$ and $\Sigma_2(B^4, F')$ are exotic.*

Finally, we give a proof of Theorem 1.9.

Proof. (of Theorem 1.9 Take the 2^n possibilities of boundary summing D and D' n times. These are topologically isotopic relative boundary, because D and D' are. Let's take two of these boundary connected sums and call them D_1 and D_2. Notice that they must differ in some position, say D_1 has a D in position i while D_2 has a D' in position i. Consider the annuli A_1 and A_2 obtained by attaching the previous band to the disk in which they differ (See figure below). Suppose D_1 and D_2 are also smoothly isotopic relative boundary. Then, so are A_1 and A_2. However, by the previous argument, $\Sigma_2(B^4, A_2)$ contains a smoothly embedded 2-sphere of square -2, while $\Sigma_2(B^4, A_1)$ does not, contradiction.

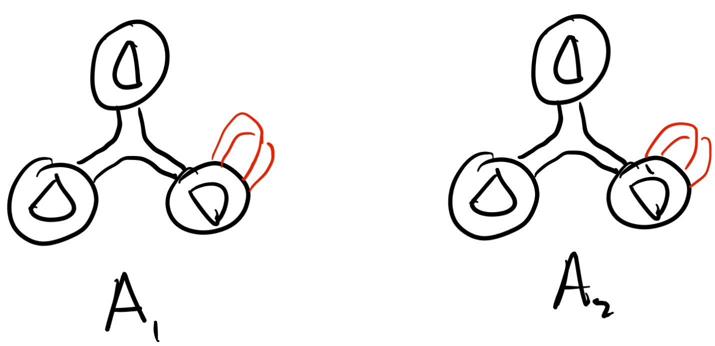

FIGURE 25. The annuli A_1 and A_2.

□

Appendix

Here we present an isotopy to go from the Kirby diagram of Figure 8 to the Kirby diagram of Figure 11.